MATH 565 Monte Carlo Methods

Fred J. Hickernell | Final Exam In Class | Thursday, December 11, 2025

Instructions:

*i. This test has **FIVE** question(s). Attempt all. The maximum number of points is **65**.*

ii. The time allowed is 120 minutes.

iii. This test is closed book, but you may use 4 double-sided letter-size sheets of notes.

iv. No calculators or other devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.

v. Keep at least three *significant digits in your intermediate calculations and final answers.*

vi. Show all your work to justify your answers. Answers without adequate justification will not receive credit.

vii. Off-site students may contact the instructor as directed by your syllabus.

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

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Signature *Date*

1. (15 points) Consider the integral

$$\mu = \int_{-1}^{3} \int_{-\infty}^{\infty} g(y,z) \exp(-y^2) \, \mathrm{d}y \, \mathrm{d}z.$$

(The inner integral is with respect to y and the outer one with respect to z.) Perform a reasonable variable transformation to make $\mu = \mathbb{E}[f(\boldsymbol{X})]$, where $\boldsymbol{X} \sim \mathrm{Unif}[0,1]^2$, i.e., the standard two-dimensional uniform distribution.

Answer: Let $y = \Phi^{-1}(x_1)/\sqrt{2}$ *and* $z = -1+4x_2$, *where* Φ *is the cumulative distribution function of the standard Gaussian random variable. Then,*

$$\begin{aligned}
&x_1 = \Phi(\sqrt{2}y), \quad \mathrm{d}x_1 = \frac{\sqrt{2}}{\sqrt{2\pi}} \exp\bigl(-(\sqrt{2}y)^2/2\bigr) \, \mathrm{d}y = \frac{\exp(-y^2)}{\sqrt{\pi}} \, \mathrm{d}y \\
&\qquad y \in (-\infty, \infty) \iff x_1 \in (0,1) \\
&\mathrm{d}z = 4\mathrm{d}x_2, \quad z \in [-1,3] \iff x_2 \in [-1,3] \\
&\mu = \int_{-1}^{3} \int_{-\infty}^{\infty} g(y,z) \exp(-y^2) \, \mathrm{d}y\mathrm{d}z \\
&\quad = \int_{-1}^{3} \int_{-\infty}^{\infty} g\bigl(\Phi^{-1}(x_1)/\sqrt{2}, -1+4x_2\bigr) \sqrt{\pi} \, \mathrm{d}x_1 \, 4\mathrm{d}x_2 \\
&\quad = \int_{[0,1]^2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \mathbb{E}[f(\boldsymbol{X})] \qquad \textit{for } f(\boldsymbol{x}) = 4\sqrt{\pi} g\bigl(\Phi^{-1}(x_1)/\sqrt{2}, -1+4x_2\bigr).
\end{aligned}$$

2. (13 points) Let $V_0, V_1, \ldots$ be IID random variables, and consider the sample mean,

$$\hat{\mu} = \frac{1}{n}(V_0 + \cdots + V_{n-1}).$$

Answer the following with justification:

a. (3 points) Is $\hat{\mu}$ an unbiased estimator of $\mathbb{E}(V_0)$?

Answer: Yes, because

$$\mathbb{E}(\hat{\mu}) = \frac{1}{n}\big[\mathbb{E}(V_0) + \cdots + \mathbb{E}(V_{n-1})\big] = \frac{n\mu}{n} = \mu$$

b. (5 points) Is $\hat{\mu}^2$ an unbiased estimator of $\mathbb{E}(V_0^2)$?

Answer: Not unless $n = 1$, *because according to the definition of variance*

$$\begin{aligned}
\mathbb{E}(\hat{\mu}^2) &= \mathrm{var}(\hat{\mu}) + [\mathbb{E}(\hat{\mu})]^2 \\
&= \frac{\mathrm{var}(V_0)}{n} + [\mathbb{E}(V_0)]^2 \\
&= \frac{E(V_0^2) - [\mathbb{E}(V_0)]^2}{n} + [\mathbb{E}(V_0)]^2 \\
&= \frac{E(V_0^2)}{n} + \frac{(n-1)[\mathbb{E}(V_0)]^2}{n}.
\end{aligned}$$

c. (5 points) Is $\hat{\mu}^2$ an unbiased estimator of $[\mathbb{E}(V_0)]^2$?

Answer: Not unless $\mathrm{var}(V_0) = 0$, *because we know from above that*

$$\mathbb{E}(\hat{\mu}^2) = [\mathbb{E}(V_0)]^2 + \frac{\mathrm{var}(V_0)}{n}.$$

3. (14 points) Consider the generation of samples from a Beta distribution, $\mathrm{Beta}(2,2)$, which has an *unnormalized* probability density function (PDF) of

$$\varrho(x) = \begin{cases} x(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

a. (4 points) What is the *normalized* PDF for this Beta distribution?

Answer: The probability density function satisfies

$$1 = \int_0^1 c\varrho(x)\mathrm{d}x = \int_0^1 cx(1-x)\mathrm{d}x = c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = c\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{c}{6}$$

So, $c = 6$, *and the normalized PDF is defined as* $\varrho_{\mathrm{norm}}(x) = 6x(1-x)$.

b. (4 points) What is the most generous acceptance rule possible for the *acceptance-rejection method* to generate $\mathrm{Beta}(2,2)$ random samples using the standard uniform distribution, $\mathrm{Unif}[0,1]$, as the proposal distribution?

Answer: The proposal density is one, and the unnormalized $\mathrm{Beta}(2,2)$ *density has a maximum of* $1/4$*, so given a proposal* U_0 *and a decsion* U_1*, both distributed* $\mathrm{Unif}[0,1]$*, the most generous acceptance rule is to accept if*

$$U_1 \le \frac{4\varrho(U_0)}{1} = 4U_0(1-U_0).$$

c. (3 points) If a uniform random number generator gives $U_0 = 0.6$ and $U_1 = 0.4$, compute $X_0 \sim \mathrm{Beta}(2,2)$ using the *acceptance-rejection method* from the previous part. If the acceptance-rejection method yields no X_0 for this choice of U_0 and U_1, explain why.

Answer: From the previous part, we accept $X_0 = U_0 = 0.6$ *if*

$$0.4 = U_1 \le 4 \times 0.6(1-0.6) = 0.96,$$

which is true.

d. (3 points) On average what proportion of the samples from the uniform distribution are accepted as samples from $\mathrm{Beta}(2,2)$?

Answer: Since the expected value of $4U_0(1-U_0)$ *for* $U_0 \sim \mathcal{U}[0,1]$ *equals* $4/6 = 2/3$ *by part a., that is the proportion of samples that will be accepted.*

4. (11 points) Customers arrive at the bank according to $\mathrm{Exp}(1/5)$, an exponential distribution with mean five minutes. The service time for a customer is a uniform distribution on the interval 1 to 7 minutes, i.e., $\mathrm{Unif}[1,7]$. Let

$$\begin{aligned} A_j &= \text{arrival time of customer } j \sim \mathrm{Exp}(1/5) \\ S_j &= \text{service time for customer } j \text{ once he or she reaches the head of the queue} \sim \mathrm{Unif}[1,7] \\ W_j &= \text{waiting time for customer } j \text{ after entering the queue before being served} \end{aligned}$$

a. (3 points) One would like to estimate the average waiting time per customer for the first m customers entering the bank assuming an empty queue at opening time, i.e.,

$$\mu = \mathbb{E}\left[\frac{1}{m}(W_1 + \cdots + W_m)\right]$$

You run the queueng simulation n times based on IID arrival times A_{ij} and service times, S_{ij}, and compute W_{ij}, where i is the index of the simulation run and j denotes which customer. What is your Monte Carlo estimate for μ?

Answer:

$$\hat{\mu} = \frac{1}{mn}\sum_{i,j=0}^{nm} W_{ij}$$

b. (4 points) Are W_{3j} and W_{7j} independent? Explain why.

Answer: Yes, because these are two independent simulation runs for the same customer.

c. (4 points) Are W_{i6} and W_{i9} independent? Explain why.

Answer: No, because the arrival and service times for customer 6 *affects the waiting time for customer* 9.

5. (12 points) An integration lattice sequence is defined by $\{\boldsymbol{x}_i = \boldsymbol{\zeta}\phi_2(i) \bmod \mathbf{1}\}_{i=0}^{\infty}$, where $\boldsymbol{\zeta}$ is an integer generator vector, and ϕ_2 is the van der Corput sequence in base 2. Considering only the first 8 points, i.e., $i = 0, \ldots, 7$ use an eyeball test to determine whether $\boldsymbol{\zeta} = (1,1)$, $(1,2)$, or $(1,3)$ gives the most evenly spread set of points on $[0,1]^2$.

Answer: We compute the first eight points

		$\boldsymbol{x}_i$ *for* $\boldsymbol{\zeta} =$		
i	$\phi_2(i)$	$(1,1)$	$(1,2)$	$(1,3)$
$0 = 000_2$	${}_20.000 = 0$	$(0,0)$	$(0,0)$	$(0,0)$
$1 = 001_2$	${}_20.100 = 1/2$	$(1/2,1/2)$	$(1/2,0)$	$(1/2,1/2)$
$2 = 010_2$	${}_20.010 = 1/4$	$(1/4,1/4)$	$(1/4,1/2)$	$(1/4,3/4)$
$3 = 011_2$	${}_20.110 = 3/4$	$(3/4,3/4)$	$(3/4,1/2)$	$(3/4,1/4)$
$4 = 100_2$	${}_20.001 = 1/8$	$(1/8,1/8)$	$(1/8,1/4)$	$(1/8,3/8)$
$5 = 101_2$	${}_20.101 = 5/8$	$(5/8,5/8)$	$(5/8,1/4)$	$(5/8,7/8)$
$6 = 110_2$	${}_20.011 = 3/8$	$(3/8,3/8)$	$(3/8,3/4)$	$(3/8,1/8)$
$7 = 111_2$	${}_20.111 = 7/8$	$(7/8,7/8)$	$(7/8,3/4)$	$(7/8,5/8)$

The choice $\boldsymbol{\zeta} = (1,1)$ *places points along the diagonal. The choice* $\boldsymbol{\zeta} = (1,2)$ *only gives* 4 *different values of* x_2. *The choice* $\boldsymbol{\zeta} = (1,3)$ *places points relatively evenly.*