MATH 563 Mathematical Statistics

Fred J. Hickernell — Final Exam — Tuesday, May 5, 2026

Q	Score
1	
2	
3	
4	
Total	

Instructions:

i. *This test has **FOUR** question(s). Attempt all. The maximum number of points is **100**.*

ii. *The time allowed is 120 minutes.*

iii. *This test is closed book, but you may use **8** double-sided letter-size sheets of notes.*

iv. *No calculators or other electronic devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.*

v. *You are expected to simplify any expressions that can be simplified as whole numbers, e.g.,* $36/15 = 12/5$*, but you may leave other expressions as is, e.g.,* $\sqrt{47}$ *or* $18/29$.

vi. *You will be provided all of the critical values that you need.*

vii. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

viii. *Print your name clearly on each answer sheet.*

ix. *Off-site students may contact the instructor as directed by your syllabus.*

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

______________________________ ____________________

Signature *Date*

____

Printed Name

Upper quantiles: q_α **satisfies** $\mathbb{P}(X > q_\alpha) = \alpha$

X	α	0.995	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
$\text{Norm}(0,1)$	z_α	-2.58	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33	2.58
F_{d_1,d_2}	$F_{1,99;\alpha}$	3.95e-05	0.000158	0.000987	0.00395	0.0159	2.76	3.94	5.18	6.90	8.24
	$F_{1,100;\alpha}$	3.95e-05	0.000158	0.000987	0.00395	0.0159	2.76	3.94	5.18	6.90	8.24
	$F_{2,99;\alpha}$	0.00501	0.0101	0.0253	0.0513	0.105	2.36	3.09	3.83	4.83	5.59
	$F_{2,100;\alpha}$	0.00501	0.0101	0.0253	0.0513	0.105	2.36	3.09	3.83	4.82	5.59
	$F_{3,99;\alpha}$	0.0238	0.0381	0.0716	0.117	0.194	2.14	2.70	3.25	3.99	4.55
	$F_{3,100;\alpha}$	0.0238	0.0381	0.0717	0.117	0.194	2.14	2.70	3.25	3.98	4.54

1. (21 points) Determine whether each statement is true or false and provide justification. If it is false, correct it to make it true.

   a. (7 points) One may double the power of a hypothesis test that is presently 70% by increasing the sample size sufficiently.

      Answer: False. Increasing the sample size increases power, but there is no guarantee it will double; in fact, power is bounded above by 100%.

   b. (7 points) One may *not* have a null hypothesis of the form $H_0 : \overline{X} = 47$.

      Answer: True. Hypotheses must be written in terms of population parameters

   c. (7 points) For generalized linear models, the *link function*, g, defines the probability distribution of the response, Y.

      Answer: False. The link function maps the set of possible means of Y *to* $\mathbb{R}$ *so that* $g\big(\mathbb{E}(Y \mid \boldsymbol{x})\big) = \boldsymbol{x}^\top \boldsymbol{\beta}$ *for some regression coefficient* β.

2. (38 points) Let $X_1, \dots X_n \overset{\text{IID}}{\sim} \text{Bern}(p)$ represent whether a treatment for a disease is effective or not.

   a. (12 points) Derive the maximum likelihood estimators for $\mathbb{E}(X_1)$ and $\text{var}(X_1)$.

      Answer: The density for the Bernoulli distribution is $\varrho(x) = p^x(1-p)^{1-x}$, *so the likelihood function in terms of the data,* $\boldsymbol{X} = (X_1, \dots, X_n)$, *is*

$$L(p \mid \boldsymbol{X}) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^T(1-p)^{n-T}$$

      where $T = X_1 + \cdots + X_n$. *Taking the log-likelihood and setting it equal to zero gives*

$$\ell(p \mid \boldsymbol{X}) = \log\big(L(p \mid \boldsymbol{X})\big) = T\log(p) + (n-T)\log(1-p)$$

$$0 = \frac{\partial \ell(p \mid \boldsymbol{X})}{\partial p} = \frac{T}{p} - \frac{n-T}{1-p} = \frac{T(1-p) - (n-T)p}{p(1-p)} = \frac{T - np}{p(1-p)}$$

$$\implies \widehat{p}_{\text{MLE}} = \frac{T}{n} = \overline{X}, \quad \textit{where } \overline{X} = \frac{1}{n}(X_1 + \cdots + X_n).$$

      Since $\mathbb{E}(X_1) = p$ *and* $\text{var}(X_1) = p(1-p)$, *it follows that* $\mu_{\text{MLE}} = \overline{X}$ *and* $\widehat{\text{var}}(X_1) = \overline{X}(1-\overline{X})$.

   b. (10 points) Are these maximum likelihood estimators for $\mathbb{E}(X_1)$ and $\text{var}(X_1)$ unbiased? Asymptotically unbiased?

      Answer: The sample mean is always an unbiased estimator of the population mean, so μ_{MLE} *is an unbiased estimator of* $\mathbb{E}(X_1)$.

However,

$$\mathbb{E}[\widehat{\mathrm{var}}(X_1)] = \mathbb{E}[\overline{X}(1-\overline{X})] = \mathbb{E}[\overline{X}] - \mathbb{E}[\overline{X}^2]$$
$$= p - \left[\mathrm{var}(\overline{X}) + \{\mathbb{E}(\overline{X})\}^2\right] = p - \frac{\mathrm{var}(X_1)}{n} - p^2$$
$$= p(1-p)\left[1 - \frac{1}{n}\right],$$

so $\widehat{\mathrm{var}}(X_1)$ *is a biased but asymptotically unbiased estimator of* $\mathrm{var}(X_1)$.

c. (6 points) The company does not want to market this treatment unless it is effective on at least half the patients. What hypothesis test should you perform to provide compelling evidence that the treatment is effective?

Answer:

$$H_0 : p \le 0.5, \quad H_A : p > 0.5$$

d. (10 points) If you observe that the treatment is effective for 64 out of 100 patients, what is your conclusion?

Answer: The test statistic under the CLT approximation and the null hypothesis is

$$Z = \frac{\overline{X} - p}{\sqrt{p(1-p)/n}} = \frac{\overline{X} - 0.5}{\sqrt{0.5(1-0.5)/100}} = \frac{\overline{X} - 0.5}{1/20} = 20(\overline{X} - 0.5) \mathrel{\dot\sim} \mathrm{Norm}(0,1).$$

The observed value of this statistic is $z = 20(0.64 - 0.5) = 2.8$. *This is greater than the critical value* $z_{0.05} = 1.64$*, so we reject the null hypothesis that the treatment is not sufficiently effective.*

3. (25 points) Golfers are randomly split into groups, and each group hits drives with a different brand of driver (golf club). Let y_{ij} denote the length in yards of the j^{th} drive with the i^{th} brand of driver. You find that

Group (driver) $i =$	1	2	3
Sample size $n_i =$	30	38	35
Group means $\overline{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$	——	——	——

$n = n_1 + n_2 + n_3 = 103$

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{3}\sum_{j=1}^{n_i} y_{ij}$$
$$\mathrm{SST} = \sum_{i=1}^{3}\sum_{j=1}^{n_i} (y_{ij} - \overline{y})^2 = 2700$$
$$\mathrm{SSW} = \sum_{i=1}^{3}\sum_{j=1}^{n_i} (y_{ij} - \overline{y}_i)^2 = 2500$$

a. (20 points) Is there compelling evidence at the 5% significance level that the average lengths of drives with different drivers are not all the same?

Answer: Let's make an ANOVA table and fill in the blanks:

Source	*df*	*Sum of Squares*	*Mean Square*	*F*
Between groups	2	200	$\frac{200}{2} = 100$	$\frac{100}{25} = 4$
Within groups	100	2500	$\frac{2500}{100} = 25$	
Total	102	2700		

Since the observed F is greater than $F_{2,100;0.05} = 3.09$, there is strong evidence that the average drive lengths are not all the same.

b. (5 points) Of all the data provided above, which are needed to arrive at your conclusion, and which is not needed?

Answer: We need k (the number of groups), n, SST, and SSW; the group means and individual n_i are not needed once sums of squares are given.

The original version of this problem had values listed for the group means, which were incompatible with SSB = SST − SSW.

4. (16 points) A covariance kernel, K, in Gaussian process regression must satisfy

Symmetry: $K(\boldsymbol{t}, \boldsymbol{x}) = K(\boldsymbol{x}, \boldsymbol{t})$ for all $\boldsymbol{t}, \boldsymbol{x}$, and

Positive definiteness: $\sum_{i,j=1}^{n} c_i c_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) > 0$ for all $\boldsymbol{c} = (c_1, \ldots, c_n) \neq \boldsymbol{0}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ distinct.

a. (5 points) Why must $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be distinct for the second condition to have a chance of holding for all $\boldsymbol{c} \neq \boldsymbol{0}$?

Answer: If they are not distinct, e.g, $\boldsymbol{x}_2 = \boldsymbol{x}_1$, then for $\boldsymbol{c} = (1, -1, 0, \ldots, 0)$ it follows that

$$\begin{aligned}\sum_{i,j=1}^{n} c_i c_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) &= K(\boldsymbol{x}_1, \boldsymbol{x}_1) - K(\boldsymbol{x}_1, \boldsymbol{x}_2) - K(\boldsymbol{x}_2, \boldsymbol{x}_1) + K(\boldsymbol{x}_2, \boldsymbol{x}_2) \\ &= K(\boldsymbol{x}_1, \boldsymbol{x}_1) - K(\boldsymbol{x}_1, \boldsymbol{x}_1) - K(\boldsymbol{x}_1, \boldsymbol{x}_1) + K(\boldsymbol{x}_1, \boldsymbol{x}_1) = 0\end{aligned}$$

b. (11 points) If K_1 and K_2 are two kernels satisfying the above conditions, i.e., symmetric and positive definite, and $K(\boldsymbol{t}, \boldsymbol{x}) = a_1 K_1(\boldsymbol{t}, \boldsymbol{x}) + a_2 K_2(\boldsymbol{t}, \boldsymbol{x})$, then under what general sufficient conditions is K also symmetric and positive definite?

Answer: The symmetry is automatic:

$$\begin{aligned}K(\boldsymbol{t}, \boldsymbol{x}) &= a_1 K_1(\boldsymbol{t}, \boldsymbol{x}) + a_2 K_2(\boldsymbol{t}, \boldsymbol{x}) \\ &= a_1 K_1(\boldsymbol{x}, \boldsymbol{t}) + a_2 K_2(\boldsymbol{x}, \boldsymbol{t}) \qquad \textit{because } K_1 \textit{ and } K_2 \textit{ are symmetric} \\ &= K(\boldsymbol{x}, \boldsymbol{t})\end{aligned}$$

The positive definite requires more. Note that

$$\sum_{i,j=1}^{n} c_i c_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) = a_1 \sum_{i,j=1}^{n} c_i c_j K_1(\boldsymbol{x}_i, \boldsymbol{x}_j) + a_2 \sum_{i,j=1}^{n} c_i c_j K_2(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

Each sum on the right is positive for all $\mathbf{c} = (c_1, \ldots, c_n) \neq \mathbf{0}$ *and* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ *distinct. Thus, positive definiteness holds if* a_1 *and* a_2 *are non-negative and at least one is positive.*

Final Exam Scores

Number of Students: 16, Minimum: 31, Maximum: 98, Mean: 66.2, Median: 72
Standard Deviation: 22.5, Quartiles (Q1, Q3): (45, 84)

9	38
8	23444
7	7
6	17
5	5
4	37
3	146