MATH 563 Mathematical Statistics

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Instructions:

i. *This test has **THREE** question(s). Attempt all. The maximum number of points is **100**.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *No calculators or other electronic devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.*

v. *You are expected to simplify any expressions that can be simplified as whole numbers, e.g.,* $36/15 = 12/5$, *but you may leave other expressions as is, e.g.,* $\sqrt{47}$ *or* $18/29$.

vi. *You will be provided all of the critical values that you need.*

vii. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

viii. *Print your name clearly on each answer sheet.*

ix. *Off-site students may contact the instructor as directed by your syllabus.*

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

______________________________ ____________________

Signature *Date*

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Printed Name

Upper quantiles: q_α **satisfies** $\mathbb{P}(X > q_\alpha) = \alpha$

X	α	0.999	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.001
Norm(0, 1)	z_α	-3.09	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33	3.09

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1. (28 points) Determine whether or not each of the following statements is true. If a statement is false, correct it to make it true.

   a. (7 points) The outcome of a hypothesis test is either to accept or reject the null hypothesis.

      Answer: False. We either reject or do not reject the null hypothesis.

   b. (7 points) The null hypothesis, H_0, and the alternative hypothesis, H_A, must be crafted so that both cannot be true simultaneously.

      Answer: True.

c. (7 points) For the estimator of the regression coefficient, $\widehat{\boldsymbol{\beta}} = (\mathsf{X}^T\mathsf{X})^{-1}\mathsf{X}^T\boldsymbol{Y}$, to be unbiased, we do not need the vector of errors, $\boldsymbol{\varepsilon}$ to be normally distributed, but they should satisfy $\mathbb{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$.

Answer: True. As some pointed out, they do not need to satisfy $\text{cov}(\boldsymbol{\varepsilon}) = \sigma^2\mathsf{I}$*, which was in the original statement of the problem, so I gave credit for both "True" and "False", provided that the correction for "False" was provided.*

d. (7 points) A confidence interval and a prediction interval are different names for the same object.

Answer: False. A confidence interval often tells us about the mean of a random variable, but a prediction interval tells us about where the next value may lie.

2. (45 points) Your boss comes to you with data and a problem. Your company has been reporting to the board that the mean lifetimes of the belts you produce are 1000 hours, but the data from $n = 64$ IID observations shows a mean of $\overline{x} = 1100$ hours and a standard deviation of 80 hours.

a. (20 points) The company board does not want to underreport or overreport the average lifetime of the belts. Is this data compelling enough to reject the hypothesis that the average lifetime is actually 1000 hours and not significantly more or less than 1000 at the $\alpha = 5\%$ significance level?

Answer: Let $H_0 : \mu = 1000$ *and* $H_A : \mu \neq 1000$*. Given the large sample size, we can use a test based on the Central Limit Theorem and replace* σ *by* $s = 80$*. One way is to compute a test statistic:*

$$T = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \approx \frac{\overline{X} - 1000}{80/\sqrt{64}} = \frac{\overline{X} - 1000}{10} \overset{\cdot}{\sim} \mathrm{N}(0,1)$$

We observe

$$t = \frac{\overline{x} - 1000}{10} = \frac{1100 - 1000}{10} = 10,$$

which is far beyond the critical value of 1.96*. We reject the* H_0*.*
Another way is to note that

$$\begin{aligned}
\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \overset{\cdot}{\sim} \mathrm{N}(0,1) \implies 5\% &\approx \mathbb{P}\left(\left|\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2}\right) \\
&= \mathbb{P}\big(\overline{X} \leq \mu - z_{\alpha/2}\sigma/\sqrt{n} \text{ or } \overline{X} \geq \mu + z_{\alpha/2}\sigma/\sqrt{n}\big) \\
&\approx \mathbb{P}\left(\overline{X} \leq 1000 - 1.96 \times 80/\sqrt{64} \text{ or } \overline{X} \geq 1000 + 1.96 \times 80/\sqrt{64}\right) \\
&= \mathbb{P}\left(\overline{X} \leq 1000 - 19.6 \text{ or } \overline{X} \geq 1000 + 19.6\right) \\
&\approx \mathbb{P}\left(\overline{X} \leq 980 \text{ or } \overline{X} \geq 1020\right).‘
\end{aligned}$$

So the rejection region for $\overline{X}$ *is* $(-\infty, 980] \cup [1020, \infty)$*. Since* $\overline{X}$ *falls inside this region, we reject the hypothesis that* $\mu = 1000$*.*

b. (10 points) What can you say about the (approximate) p-value corresponding to $\overline{x} = 1100$? You may use the notation that Φ is the CDF of the standard normal distribution.

Answer: The p-value is the probability of getting an event as extreme as $\overline{x} = 1100$ *assuming that the null hypothesis is true, so*

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \dot{\sim} \mathrm{N}(0,1) \implies \mathbb{P}\left(\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| \geq \left|\frac{\overline{x}-\mu}{\sigma/\sqrt{n}}\right|\right) \approx \mathbb{P}\left(\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| \geq \left|\frac{1100-1000}{80/\sqrt{64}}\right|\right)$$
$$= \mathbb{P}\left(\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| \geq \frac{100}{10} = 10\right) = 2\Phi(-10),$$

and the p-value is very tiny. In fact it is $< 10^{-20}$.

c. (15 points) What is the (approximate) power of the test that you developed above as a function of μ_1, the true mean? Again, you may use the notation that Φ is the CDF of the standard normal distribution.

Answer: The power is the probability of rejecting H_0 *if* $\mu = \mu_1$, *so*

$$\begin{aligned}
\text{power}(\mu_1) &\approx \mathbb{P}\left(\left|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2} \mid \mu = \mu_1\right) \\
&= \mathbb{P}\left(\left|\frac{\overline{X}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}}\right| \geq z_{\alpha/2} \mid \mu = \mu_1\right) \\
&= \mathbb{P}\left(\frac{\overline{X}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \leq -z_{\alpha/2} \text{ or } \frac{\overline{X}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \geq z_{\alpha/2} \mid \mu = \mu_1\right) \\
&= \mathbb{P}\left(\frac{\overline{X}-\mu_1}{\sigma/\sqrt{n}} \leq -z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \text{ or } \frac{\overline{X}-\mu_1}{\sigma/\sqrt{n}} \geq z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_1\right) \\
&= \Phi\left(-z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}}\right) + \left[1 - \Phi\left(z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}}\right)\right] \\
&= \Phi\left(-1.96 + \frac{1000-\mu_1}{10}\right) + \left[1 - \Phi\left(1.96 - \frac{\mu_1-1000}{10}\right)\right]
\end{aligned}$$

If μ_1 *is significantly smaller than 1000, then the first term approaches one. If* μ_1 *is significantly larger than* 1000, *the second term approaches one. In either case, the power would approach one.*

3. (27 points) Suppose that $X_1, X_2 \overset{\text{IID}}{\sim} \text{Exp}(\lambda)$. Show that $T(X_1, X_2) := X_1 + X_2$ is a minimal sufficient statistic for λ, but that $\boldsymbol{T}'(X_1, X_2) := (X_1, X_2)$ is not.

Answer: The likelihood and the log likelihood given data (x_1, x_2) *are*

$$L\big(\lambda \mid (x_1, x_2)\big) = \lambda \exp(-\lambda x_1)\lambda \exp(-\lambda x_2) = \lambda^2 \exp\big(-\lambda(x_1 + x_2)\big)$$
$$\ell\big(\lambda \mid (x_1, x_2)\big) = \log\left(L\big(\lambda \mid (x_1, x_2)\big)\right) = 2\log(\lambda) - \lambda(x_1 + x_2) = 2\log(\lambda) - \lambda T(x_1, x_2)$$

Given any data $\boldsymbol{x} = (x_1, x_2)$ *and* $\boldsymbol{y} = (y_1, y_2)$

$$\ell(\lambda \mid \boldsymbol{x}) - \ell(\lambda \mid \boldsymbol{y}) = -\lambda[T(\boldsymbol{x}) - T(\boldsymbol{y})],$$

So, $\ell(\lambda \mid \boldsymbol{x}) - \ell(\lambda \mid \boldsymbol{y})$ *independent of* λ *iff* $T(\boldsymbol{x}) = T(\boldsymbol{y})$*, so* $T(X_1, X_2) = X_1 + X_2$ *is minimal sufficient.*

On the other hand, take $\boldsymbol{x} = (1, 2)$ *and* $\boldsymbol{y} = (2, 1)$*. Then*

$$T(\boldsymbol{x}) = T(\boldsymbol{y}) = 3,$$

so $\ell(\lambda \mid \boldsymbol{x}) - \ell(\lambda \mid \boldsymbol{y})$ *is independent of* λ*. However,*

$$\boldsymbol{T}'(\boldsymbol{x}) = (1, 2) \neq (2, 1) = \boldsymbol{T}'(\boldsymbol{y}).$$

Therefore $\boldsymbol{T}'$ *is sufficient but not minimal sufficient.*

Test 2 Scores (Regular and Make-Up)
Number of Students: 16, Minimum: 14, Maximum: 98, Mean: 58.9, Median: 58
Standard Deviation: 23.2, Quartiles (Q1, Q3): (41.5, 76.5)

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9|8
8|35
7|3467
6|0
5|16
4|47
3|79
2|9
1|4
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