MATH 563 Mathematical Statistics

Fred J. Hickernell — Test 2 Make Up — Tuesday, April 17, 2026

Instructions:

i. *This test has **FOUR** question(s). Attempt all. The maximum number of points is **100**.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *No calculators or other electronic devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.*

v. *You are expected to simplify any expressions that can be simplified as whole numbers, e.g.,* $36/15 = 12/5$, *but you may leave other expressions as is, e.g.,* $\sqrt{47}$ *or* $18/29$.

vi. *You will be provided all of the critical values that you need.*

vii. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

viii. *Print your name clearly on each answer sheet.*

ix. *Off-site students may contact the instructor as directed by your syllabus.*

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

______________________________ *Signature*

______________________________ *Date*

______________________________ *Printed Name*

Upper quantiles: q_α **satisfies** $\mathbb{P}(X > q_\alpha) = \alpha$

X	α	0.999	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.001
$\text{Norm}(0, 1)$	z_α	-3.09	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33	3.09

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1. (21 points) Determine whether or not each of the following statements is true for the hypothesis test $H_0 : \mu = 47$ and $H_A : \mu > 47$. If a statement is false, correct it to make it true.

   a. (7 points) It is possible to reject H_0, even if one observes $\overline{x} = 45$.

   Answer: False. One must have $\overline{x} > 47$ *to reject the null hypothesis.*

   b. (7 points) It is possible to *not* reject H_0, even if one observes $\overline{x} = 50$.

   Answer: True. It depends on the n *and perhaps on* S *to determine whether* H_0 *will be rejected.*

c. (7 points) It is possible to reject H_0 even if $\mu = 47$.

Answer: True, with Type 1 error α.

2. (29 points) Your coach tells you to work on your lay-ups. She claims that you only make 50%. You want to prove her wrong at the $\alpha = 5\%$ significance level, and you make 5 layups in a row.

a. (20 points) Is this compelling enough evidence to prove your coach wrong?

Answer: The null hypothesis is $H_0 : p = 0.5$, and the alternative is $H_A : p > 0.5$. Under the null hypothesis, the probability of making 5 layups in a row is $0.5^5 = 1/32$, which is less than 5%. Therefore, you have evidence to reject your coach's claim.

b. (9 points) If you want compelling evidence at the $\alpha = 1\%$, how many layups would you need to make in a row?

Answer: Now we need $0.5^n < 0.01$ so $n = 7$. Seven layups in a row. Otherwise you might just have been lucky.

3. (30 points) Let $\overline{X}$ be the sample mean of IID exponential random data with rate $\lambda = 1/\mu$. Show that $\overline{X}$ attains the Cramer Rao lower bound for variance of an unbiased estimator for μ.

Answer: The log likelihood is

$$\begin{aligned}
\ell(\lambda \mid \boldsymbol{X}) &= \log\left(\prod_{i=1}^{n} \lambda \exp(-\lambda x_i)\right) \\
&= n\log(\lambda) - \lambda \sum_{i=1}^{n} x_i = n\log(\lambda) - n\lambda\overline{X} \\
\ell(\mu \mid \overline{X}) &= n\left[-\log(\mu) - \frac{\overline{X}}{\mu}\right] \\
\frac{\partial \ell(\mu \mid \overline{X})}{\partial \mu} &= n\left[-\frac{1}{\mu} + \frac{\overline{X}}{\mu^2}\right] \\
\frac{\partial^2 \ell(\mu \mid \overline{X})}{\partial \mu^2} &= n\left[\frac{1}{\mu^2} - \frac{2\overline{X}}{\mu^3}\right] \\
I(\mu) &= -\mathbb{E}\left(\frac{\partial^2 \ell(\mu \mid \overline{X})}{\partial \mu^2}\right) = -n\left[\frac{1}{\mu^2} - \frac{2\mu}{\mu^3}\right] = \frac{n}{\mu^2}
\end{aligned}$$

Thus, for any Θ that is unbiased estimator of μ

$$\operatorname{var}(\Theta) \geq \frac{1}{I(\mu)} = \frac{\mu^2}{n} = \frac{\operatorname{var}(X)}{n} = \operatorname{var}(\overline{X}).$$

4. (20 points) Let $\boldsymbol{Y}$ be a vector of response data, X be the design matrix, and $\widehat{\boldsymbol{\beta}}$ be the estimated coefficients for an ordinary least squares regression model. Assume that the errors are IID with zero mean and variance σ^2. Furthermore, suppose that X is designed so that the columns of X are orthonormal.

   a. (15 points) What is the covariance matrix for $\widehat{\boldsymbol{\beta}}$ under this simplification?

      Answer: Since

$$\widehat{\boldsymbol{\beta}} = (\mathsf{X}^T\mathsf{X})^{-1}\mathsf{X}^T\boldsymbol{Y} = \mathsf{X}^T(\mathsf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) = \beta + \mathsf{X}^T\boldsymbol{\varepsilon}$$

      It follows that

$$\text{cov}(\widehat{\boldsymbol{\beta}}) = \mathbb{E}[(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top] = \mathbb{E}[\mathsf{X}^T\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top\mathsf{X}] = \sigma^2\mathsf{X}^T\mathsf{X} = \sigma^2\mathsf{I}$$

   b. (5 points) In particular, what is $\text{corr}(\widehat{\beta}_1, \widehat{\beta}_2)$?

      Answer: Zero