MATH 563 Mathematical Statistics

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Instructions:

i. *This test has **FOUR** question(s). Attempt all. The maximum number of points is **100**.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *No calculators or other electronic devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.*

v. *You are expected to simplify any expressions that can be simplified as whole numbers, e.g.,* $36/15 = 12/5$*, but you may leave other expressions as is, e.g.,* $\sqrt{47}$ *or* $18/29$.

vi. *You will be provided all of the critical values that you need.*

vii. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

viii. *Print your name clearly on each answer sheet.*

ix. *Off-site students may contact the instructor as directed by your syllabus.*

Q	**Score**
1	
2	
3	
4	
Total	

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

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Signature

____
Date

____
Printed Name

Upper quantiles: q_α **satisfies** $\mathbb{P}(X > q_\alpha) = \alpha$

X	α	0.995	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
$\text{Norm}(0,1)$	z_α	-2.58	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33	2.58
χ^2_ν	$\chi^2_{10,\alpha}$	2.16	2.56	3.25	3.94	4.87	16.0	18.3	20.5	23.2	25.2
	$\chi^2_{20,\alpha}$	7.43	8.26	9.59	10.9	12.4	28.4	31.4	34.2	37.6	40.0
	$\chi^2_{39,\alpha}$	20.0	21.4	23.7	25.7	28.2	50.7	54.6	58.1	62.4	65.5
	$\chi^2_{40,\alpha}$	20.7	22.2	24.4	26.5	29.1	51.8	55.8	59.3	63.7	66.8
	$\chi^2_{41,\alpha}$	21.4	22.9	25.2	27.3	29.9	52.9	56.9	60.6	65.0	68.1

For problems 1. and 2. we have random data, $X_1, \ldots, X_n \overset{\text{IID}}{\sim} \text{Exp}(\lambda)$**, where** $\lambda = 1/\mathbb{E}(X_1)$**.**

1. (40 points)

a. (15 points) What is the maximum likelihood estimator (MLE) for the parameter λ?

Answer: Let $\boldsymbol{X} = (X_1, \dots, X_n)$. Since the likelihood and log likelihood for the data are

$$L(\lambda \mid \boldsymbol{X}) = \varrho_{\boldsymbol{X}}(\boldsymbol{X} \mid \lambda) = \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) = \lambda^n \exp(-\lambda T_n), \qquad T_n := X_1 + \cdots X_n$$
$$\ell(\lambda \mid \boldsymbol{X}) = \log\bigl(L(\lambda \mid \boldsymbol{X})\bigr) = n \log(\lambda) - \lambda T_n$$

where $\varrho_{\boldsymbol{X}}$ is the multivariate density of the data. The MLE for λ comes from maximizing the log likelihood:

$$0 = \frac{\partial \ell(\lambda \mid \boldsymbol{x})}{\partial \lambda} \implies 0 = \frac{n}{\lambda} - T_n \implies \Lambda_{MLE,n} = \frac{n}{T_n} = \frac{1}{\overline{X}_n}$$

where $\overline{X}_n$ is the sample mean.

b. (15 points) What is the MLE for $\mu = \mathbb{E}(X_1)$? Is it unbiased?

Answer: Since $\mu = 1/\lambda$, the MLE for μ is $\hat{\mu}_{MLE,n} = 1/\Lambda_{MLE,n} = \overline{X}_n$. It is unbiased because

$$\mathbb{E}(\overline{X}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{1}{n}(n\mu) = \mu.$$

c. (10 points) What is MLE for σ, the standard deviation of X_1? Is it unbiased?

Answer: Since for $\mathrm{Exp}(\lambda)$ it follows that $\sigma = 1/\lambda = \mu$, its MLE is the same as that for μ, $S_{MLE,n} = \overline{X}_n$, and so it is also unbiased.

This is a standard problem about MLE.

2. (30 points)

a. (15 points) Construct a one-sided 95% confidence interval for μ of the form $[0, m_u]$, assuming $n = 10$ and $\overline{x}_{10} = 14.5$.

Answer: Because $2n\lambda\overline{X}_n \sim \chi^2_{2n}$, *we have*

$$95\% = \mathbb{P}\left(\chi^2_{2n,0.95} \leq 2n\lambda\overline{X}_n < \infty\right) = \mathbb{P}\left(0 < \mu = \frac{1}{\lambda} \leq \frac{2n\overline{X}_n}{\chi^2_{2n,0.95}}\right)$$

For this data

$$m_u = \frac{20 \times \overline{x}_{10}}{\chi^2_{2n,0.95}} = \frac{290}{10.9} = 26.6.$$

b. (15 points) Under a large sample size, n, approximation, how large does n need to be to construct a 95% confidence interval for μ of the form $[0, m_u]$, where m_u is no more than 20% larger than $\overline{x}_n$?

Answer: For large sample size we have

$$95\% \approx \mathbb{P}\left(0 \leq \mu \leq \overline{X}_n + z_{0.05}\frac{\overline{X}_n}{\sqrt{n}}\right) = \mathbb{P}\left(0 \leq \mu \leq \overline{X}_n\left[1 + \frac{1.64}{\sqrt{n}}\right]\right),$$

since $\sigma = \mu \approx \overline{X}_n$. *To make* $m_u \leq 1.2\overline{x}$, *we need*

$$\frac{1.64}{\sqrt{n}} \leq 0.2 \implies n \geq \left(\frac{1.64}{0.2}\right)^2 = (8.2)^2 \geq 68$$

You need to recognize that part a requires a small sample size confidence interval using the χ^2 distribution. Also, this is a one-sided confidence interval. The critical value corresponds to 5%, not 2.5%. For the exponential distribution, we can approximate the standard deviation by the sample mean. For part b. you need to understand what is being asked.

3. (18 points) The manufacturer wants to construct a 99% confidence interval of the form $[0, s_u]$ for σ, the standard deviation of the length of a part. Assuming that the part length is distributed $\mathrm{N}(\mu, \sigma^2)$, and the sample standard deviation from 40 IID samples is 0.015, construct such a confidence interval.

Answer: If S^2 denotes the random observed sample variance, then

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1} \implies 99\% = \mathbb{P}\left(S_{40}^2 \geq \frac{\sigma^2 \chi^2_{39,0.99}}{39} = \frac{21.4\sigma^2}{39}\right) = \mathbb{P}\left(\sigma \leq S_{40}\sqrt{\frac{39}{21.4}}\right)$$

so for this data with $s = 0.015$,

$$s_u = 0.015\sqrt{\frac{39}{21.4}} = 0.0202.$$

It is important to understand that not all confidence interval are for means and tha in this case one needs $\chi^2_{39,0.99}$, not $\chi^2_{39,0.01}$.

4. (12 points) Answer each of the following questions in a couple of sentences.

a. (4 points) Can a discrete random variable, X, have a mean, μ, for which $\mathbb{P}(X = \mu) = 0$?

Answer: Yes. E.g., $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0.5$ and $\mu = 0.5$.

b. (4 points) What kind of confidence interval should be used to measure the effectiveness of a class when one can compare the performance of the same students taking a test at the beginning of the term and another test at the end of the term?

Answer: This is a mean of differences, the difference in the score for a student.

c. (4 points) If IID data with common mean μ and variance σ^2 have sample mean $\overline{X}_n$, and if $Z \sim \mathrm{N}(0, 1)$, can we say that

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{\text{a. s.}} Z \text{ as } n \to \infty?$$

Answer: No. The CLT gives convergence in distribution to $Z \sim \mathrm{N}(0, 1)$, but this does not imply almost sure convergence. Indeed, the standardized mean does not converge almost surely; it continues to fluctuate as $n \to \infty$.

Part a. emphasizes an important point about means. Part c. is deceptively like the CLT.

Test 1 Scores

Number of Students: 16, Minimum: 7, Maximum: 77, Mean: 40.3, Median: 40

Standard Deviation: 25.6, Quartiles (Q1, Q3): (13.5, 63)

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7|127
6|8
5|488
4|4
3|46
2|2
1|34
0|789
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