MATH 563 Mathematical Statistics

Fred J. Hickernell — Test 1 Make Up — Thursday, March 27, 2026

Instructions:

i. *This test has **FOUR** question(s). Attempt all. The maximum number of points is **100**.*

ii. *The time allowed is 75 minutes.*

iii. *This test is closed book, but you may use 4 double-sided letter-size sheets of notes.*

iv. *No calculators or other electronic devices are allowed. Phones must be placed in your bags under your desks or at the front of the room. Hands must be on top of your desks.*

v. *You are expected to simplify any expressions that can be simplified as whole numbers, e.g.,* $36/15 = 12/5$, *but you may leave other expressions as is, e.g.,* $\sqrt{47}$ *or* $18/29$.

vi. *You will be provided all of the critical values that you need.*

vii. *Show all your work to justify your answers. Answers without adequate justification will not receive credit.*

viii. *Print your name clearly on each answer sheet.*

ix. *Off-site students may contact the instructor as directed by your syllabus.*

I understand these instructions and have not relied on any help for this exam beyond what is allowed, nor provided any help to anyone else beyond what is allowed.

______________________________ ____________________

Signature *Date*

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Printed Name

Upper quantiles: q_α **satisfies** $\mathbb{P}(X > q_\alpha) = \alpha$

X	α	0.995	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
$\text{Norm}(0,1)$	z_α	-2.58	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33	2.58
t_ν	$t_{15,\alpha}$	-2.95	-2.60	-2.13	-1.75	-1.34	1.34	1.75	2.13	2.60	2.95
	$t_{16,\alpha}$	-2.92	-2.58	-2.12	-1.75	-1.34	1.34	1.75	2.12	2.58	2.92
	$t_{17,\alpha}$	-2.90	-2.57	-2.11	-1.74	-1.33	1.33	1.74	2.11	2.57	2.90

1. (30 points) Let $X_1, \ldots, X_n$ be IID, and let $\overline{X}_n := n^{-1} \sum_{i=1}^{n} X_i$ be the sample mean.

   a. (10 points) Give an example of a nontrivial probability distribution for which $\overline{X}_n$ is an unbiased estimator of the population mean $\mu := \mathbb{E}(X)$.

   Answer: The sample mean is an unbiased estimator of the mean for all probability distributions, e.g. $X \sim \mathrm{Exp}(\lambda)$, *where* $\mu = 1/\lambda$.

   b. (10 points) Give an example of a nontrival probability distribution for which $\overline{X}_n$ is an unbiased estimator of the standard deviation $\sigma := \sqrt{\mathbb{E}\big[(X - \mathbb{E}(X))^2\big]}$.

   Answer: Since the standard deviation for the distribution $\mathrm{Exp}(\lambda)$ *is also the mean, then* $\overline{X}_n$ *is an unbiased estimator of the standard deviation.*

   c. (10 points) Give an example of a nontrival probability distribution for which $\overline{X}_n$ is *not* an unbiased estimator of the standard deviation.

   Answer: For the distribution $\mathrm{N}(\mu, \sigma^2)$ *for any* $\sigma \neq \mu$, $\overline{X}_n$ *will not be an unbiased estimator of* σ.

2. (20 points) Let $X_1, \ldots, X_n \overset{\text{IID}}{\sim} \mathrm{Unif}[0, \theta]$. Derive the maximum likelihood estimator of θ.

   Answer: The likelihood function is

$$L(\theta \mid \{X_i\}_{i=1}^n) = \prod_{i=1}^{n} \frac{\mathbb{1}(0 \le X_i \le \theta)}{\theta} = \frac{\mathbb{1}(0 \le \max_i X_i \le \theta)}{\theta^n}$$

   The maximum of this function is obtained by choosing θ *as small as possible, so* $\Theta_{MLE} = \max_i X_i$.

3. (30 points) Consider data of the form $X_1, \ldots, X_n \sim \mathrm{N}(\mu, \sigma^2)$, where the sample mean is $\overline{X}_n$ and the sample variance is S_n^2.

   a. (15 points) Under what conditions can you be 95% confident that $\mu \ge 0$ for $n = 16$?

   Answer: Since the data is normal,

$$T = \frac{\overline{X}_{16} - \mu}{S_{16}/\sqrt{16}} \sim t_{15} \implies \mathbb{P}\left[\frac{\overline{X}_{16} - \mu}{S_{16}/4} \le t_{15,0.05}\right] = 95\%$$
$$\implies \mathbb{P}\left[\mu \ge \overline{X}_{16} - 1.75 S_{16}/4\right] = 95\%$$

   If $\overline{X}_{16} - 1.75 S_{16}/4 \ge 0$, *then we can be* 95% *confident that* $\mu \ge 0$.

b. (15 points) If $\overline{X}_n = 0.5$ for some large n, and $\text{var}(X) = 9$, about how large an n would be required to show that $\mu \geq 0$ with 95% confidence.

Answer: For large sample size we have

$$Z = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1) \implies \mathbb{P}\left[\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leq z_{0.05}\right] = 95\%$$

$$\implies \mathbb{P}\left[\mu \geq 0.5 - 1.64 \times 3/\sqrt{n}\right] = 95\%$$

$$0.5 - 1.64 \times 3/\sqrt{n} \geq 0 \implies n \geq \left(\frac{1.64 \times 3}{0.5}\right)^2 = (1.64 \times 6)^2 \approx 97$$

4. (20 points) Consider the sequence of independent random variables $X_1, X_2, \ldots$, where

$$\mathbb{P}(X_i = x) = \begin{cases} 1 - i^{-2}, & x = 0, \\ i^{-2}, & x = 1. \end{cases}$$

a. (15 points) Determine whether or not $X_i \xrightarrow{\mathbb{P}} 0$.

Answer: Since $\mathbb{P}\big(|X_i - 0| \geq \varepsilon\big) = i^{-2}$ *for* $\varepsilon < 1$, *and this vanishes as* $i \to \infty$ *, it follows that* $X_i \xrightarrow{\mathbb{P}} 0$.

b. (5 points) Compute $\mathbb{E}[X_1 + X_2 + \cdots]$.

Answer: Since $\mathbb{E}(X_i) = 0 \times \mathbb{P}(X_i = 0) + 1 \times \mathbb{P}(X_i = 1) = i^{-2}$,

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} \mathbb{E}(X_i) = \sum_{i=1}^{\infty} i^{-2} = \frac{\pi^2}{6}.$$