# MATH 332 Linear Algebra: Theory and Applications

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1. Let $P = (1, 0, 1)$, $Q = (2, 1, 1)$, $R = (1, 2, 3)$, and $S = (2, 0, 2)$. Define the three displacement vectors from $P$ by $\boldsymbol{u} = Q - P$, $\boldsymbol{v} = R - P$, and $\boldsymbol{w} = S - P$.

   a. Find a nonzero vector normal to the plane through $P, Q, R$ and an equation for that plane. Explain why the three points determine a plane.

   b. Find the area of triangle $PQR$.

   c. Find the volume of the parallelepiped with edge vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Are the four points coplanar? Justify your answer.

   d. Find the distance from $S$ to the plane through $P, Q, R$.

2. Consider two linear maps from $\mathbb{R}^3$ to $\mathbb{R}^2$, $\boldsymbol{x} \mapsto \mathsf{A}_1\boldsymbol{x}$ and $\boldsymbol{x} \mapsto \mathsf{A}_2\boldsymbol{x}$. They give the same output at each of the inputs $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$. Must the maps agree for every input, so that $\mathsf{A}_1 = \mathsf{A}_2$?

3. For real $a$, define a candidate inner product $\langle \cdot, \cdot \rangle_a$ on $\mathbb{R}^2$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_a = \boldsymbol{x}^\top \mathsf{W}_a \boldsymbol{y}, \qquad \mathsf{W}_a = \begin{bmatrix} 2 & a \\ a & 2 \end{bmatrix}.$$

   a. Determine exactly which values of $a$ make $\langle \cdot, \cdot \rangle_a$ an inner product. For every rejected range or boundary value, exhibit a nonzero vector violating positive definiteness.

   b. Fix $a = 1$. Find the orthogonal projection, in this inner product, of $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ onto $S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Verify that the residual is orthogonal to $S$ in the specified inner product.

   c. Find the distance from $\boldsymbol{b}$ to $S$ in this weighted norm. Compare the projected point with the ordinary Euclidean projection, and compare their squared distances from $\boldsymbol{b}$ in the weighted norm.

   d. Give the matrix $\mathsf{P}$ of the weighted projection. Verify $\mathsf{P}^2 = \mathsf{P}$ and $\mathsf{P}^\top \mathsf{W}_1 = \mathsf{W}_1 \mathsf{P}$. Explain how $\mathsf{P}$ can represent an orthogonal projection in this inner product even though $\mathsf{P}^\top \neq \mathsf{P}$.

4. Let

$$\mathsf{R} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

   a. Find the monic characteristic polynomial, the complex eigenvalues, and a basis for each complex eigenspace.

   b. Decide whether $\mathsf{R}$ is diagonalizable over $\mathbb{R}$ and over $\mathbb{C}$. Where diagonalization is possible, find an invertible matrix $\mathsf{V}$ and a diagonal matrix $\mathsf{D}$ with $\mathsf{R} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1}$.

   c. Describe the action on real vectors geometrically. Give a real formula for $\mathsf{R}^n$ for every integer $n \geq 0$, and find the smallest positive integer $n$ for which $\mathsf{R}^n = \mathsf{I}$.

   d. For $\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, compute $\mathsf{R}^2\boldsymbol{x}$ and its Euclidean length. Is $\mathsf{R}$ singular? Explain how your answer relates to its real eigendirections.

5. Let $\mathsf{A}$ be a nonzero $5 \times 3$ matrix. What are the smallest and largest possible dimensions of its column space $\operatorname{col}(\mathsf{A})$?

6. You are given that the real solution space of $y'' + 4y' + 4y = 0$ is

$$S = \{(a + bt)e^{-2t} : a, b \in \mathbb{R}\},$$

   with basis $\mathcal{B} = \{e^{-2t}, te^{-2t}\}$, in the listed order. Define the initial-data map $E : S \to \mathbb{R}^2$ by $E(y) = \begin{bmatrix} y(0) & y'(0) \end{bmatrix}^{\top}$.

   a. Explain why $E$ is linear. Find its matrix with input basis $\mathcal{B}$ and the standard output basis.

   b. Show that every pair of initial values determines exactly one function in $S$. Find the matrix of the inverse map $E^{-1} : \mathbb{R}^2 \to S$, using the standard input basis and $\mathcal{B}$ as the output basis.

   c. Find the function with $y(0) = 2$ and $y'(0) = -1$, and check both initial conditions directly.

   d. Show that $\mathcal{C} = \{(1 + 2t)e^{-2t}, te^{-2t}\}$ is another basis of $S$. What are the coordinates of a general $y \in S$ in this basis, in terms of its initial data? Explain why these coordinates represent a function rather than merely its value at one time.

7. The data matrix $\mathsf{X}$ has four observations as rows and two features as columns. Let

$$\mathsf{X} = \begin{bmatrix} 3 & 3 \\ -3 & -3 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathsf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}.$$

   a. Check that the data are centered and that $\mathsf{Q}$ is orthogonal. Compute $\mathsf{S} = \mathsf{X}\mathsf{Q}$; its columns are called score vectors. State the spaces containing a direction $\boldsymbol{q}_j$ and its score vector $\mathsf{X}\boldsymbol{q}_j$.

   b. The sample covariance is $\mathsf{C} = \mathsf{X}^{\top}\mathsf{X}/3$. Show that the columns of $\mathsf{Q}$ are eigenvectors of $\mathsf{C}$. Find the sample variance of each score vector and the fraction of total variance carried by the first score vector.

c. Reconstruct the first observation (the first row of $\mathsf{X}$) from its two scores (the first row of $\mathsf{S}$). Then give the approximation to all four observations obtained by discarding the second score vector. What is the sum of squared reconstruction errors?

d. Explain why the full coordinate change preserves each observation's length and the total variance. Does choosing these directions require a response variable? Explain.

8. In a two-sector Leontief economy the input-requirements matrix $\mathsf{C}$ and the external-demand vector $\boldsymbol{d}$ are

$$\mathsf{C} = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}, \qquad \boldsymbol{d} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Entry $c_{ij}$ is the amount of sector $i$ output required per unit of sector $j$ output.

a. Find the gross-production vector $\boldsymbol{x}$ and the internal demand for each sector.

b. Suppose external demand for sector 2 rises by three units and external demand for sector 1 stays fixed. How much does gross production rise in each sector? Explain why sector 1 production changes.

9. Use the standard Euclidean inner product. Let

$$\mathsf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 2 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

a. Find a $3 \times 2$ matrix $\mathsf{Q}$ with orthonormal columns and a $2 \times 2$ upper triangular matrix $\mathsf{R}$ with positive diagonal entries such that $\mathsf{A} = \mathsf{QR}$.

b. For the least-squares problem $\mathsf{A}\boldsymbol{x} \approx \boldsymbol{b}$, find the coefficient vector $\widehat{\boldsymbol{x}}$, the fitted vector $\widehat{\boldsymbol{b}} = \mathsf{A}\widehat{\boldsymbol{x}}$, and the residual $\boldsymbol{r} = \boldsymbol{b} - \widehat{\boldsymbol{b}}$. Verify the residual's orthogonality to both columns of $\mathsf{A}$.

c. Compute $\mathsf{QQ}^{\top}$. Explain why $\mathsf{Q}^{\top}\mathsf{Q} = \mathsf{I}_2$ but $\mathsf{QQ}^{\top} \neq \mathsf{I}_3$.

d. Are the two entries of $\mathsf{Q}^{\top}\boldsymbol{b}$ coordinates of $\boldsymbol{b}$ in a basis of $\mathbb{R}^3$? Explain. Extend the columns of $\mathsf{Q}$ to an ordered orthonormal basis of $\mathbb{R}^3$ and find all three coordinates of $\boldsymbol{b}$.

10. A model uses

$$f(t) = \beta_1 + \beta_2 t^2 + \beta_3(1 + t^2)$$

for the observations $(t, y) = (-1, 2), (0, 1), (1, 4)$.

a. Write the model in the form $\mathsf{X}\boldsymbol{\beta} \approx \boldsymbol{y}$, where $\mathsf{X}$ is the design matrix, $\boldsymbol{\beta}$ is the coefficient vector, and $\boldsymbol{y}$ is the observed response vector. Use the listed order of observations. Explain why this is a linear model in the coefficients even though $t^2$ occurs. Find the rank and a basis for null($\mathsf{X}$).

b. For $\widehat{\boldsymbol{\beta}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, compute the fitted response and residual $\boldsymbol{r} = \boldsymbol{y} - \mathsf{X}\widehat{\boldsymbol{\beta}}$. Determine whether these coefficients minimize the sum of squared response errors. Justify your conclusion.

c. Describe all coefficient vectors giving this same fit. Give two independent linear combinations of coefficients with fixed values across all these vectors.

d. Determine the prediction at $t = 2$. Is this prediction unique? Is exact interpolation of all three observations possible? Explain.

11. A $4 \times 4$ matrix $\mathsf{A}$ has determinant $\det(\mathsf{A}) = -3$. Add five times row 1 to row 2, then interchange rows 1 and 4. Call the resulting matrix $\mathsf{B}$. What is $\det(\mathsf{B})$?

12. For real parameters $a$ and $b$, consider the linear system

$$\mathsf{A}_a \boldsymbol{x} = \boldsymbol{c}_b, \qquad \mathsf{A}_a = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & a \end{bmatrix}, \qquad \boldsymbol{c}_b = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix}.$$

    a. Determine exactly when the system has one solution, no solution, or infinitely many solutions.
    b. Find all solutions in each consistent case.
    c. Give the rank and nullity of $\mathsf{A}_a$ for every $a$. Describe the geometry of each nonempty solution set, and explain why two solutions of the same system differ by a null-space vector.

13. Consider the supplied initial-value problem

$$\boldsymbol{x}'(t) = \mathsf{A}\boldsymbol{x}(t), \qquad \mathsf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

    a. Find an orthogonal matrix $\mathsf{Q}$ whose columns are eigenvectors of $\mathsf{A}$ and the corresponding diagonal matrix $\mathsf{D}$ of eigenvalues. Explain why such a factorization exists for this matrix.
    b. Find an explicit formula for the matrix exponential $e^{t\mathsf{A}}$. Check its value at $t = 0$.
    c. Solve the initial-value problem. Verify the initial value and the differential equation directly.
    d. Describe the behavior of the solution $\boldsymbol{x}(t)$ as $t \to \infty$. Is $e^{t\mathsf{A}}$ invertible at every finite real $t$? Justify your conclusion and explain how it is consistent with the limiting behavior.

14. What is the dimension of $\mathbb{C}^2$ when the scalar field is $\mathbb{R}$? What is its dimension when the scalar field is $\mathbb{C}$?

15. A PLU factorization of $\mathsf{A}$ is given below.

$$\mathsf{PA} = \mathsf{LU}, \qquad \mathsf{A} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & 2 & -1 \\ -1 & 4 & 7 \end{bmatrix}, \qquad \mathsf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \qquad \mathsf{U} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

    a. Solve $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}$. Show your work.
    b. Find $\det(\mathsf{A})$. Explain how the factorization also shows that every right-hand side has exactly one solution.
    c. Suppose the same $n \times n$ coefficient matrix must be used with 100 different right-hand sides. Explain what can be reused, and compare the usual arithmetic costs of a fresh factorization and one solve with saved factors. Is forming $\mathsf{A}^{-1}$ necessary?

16. Find the point on the line $(1, 0, -1) + t(1, 2, 2)$, $t \in \mathbb{R}$, closest to $(5, 5, 1)$, and find the distance from $(5, 5, 1)$ to the line.

17. For each assertion about a real $m \times n$ matrix $\mathsf{A}$, decide whether it is true. Prove a true assertion; for a false assertion give a concrete matrix and right-hand sides that establish the failure.

    a. If $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for one right-hand side $\boldsymbol{b}$, then it has exactly one solution for every right-hand side in $\mathbb{R}^m$.

    b. If $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b} \in \mathbb{R}^m$, then the columns of $\mathsf{A}$ are linearly independent.

    c. If the columns of $\mathsf{A}$ are linearly independent, then $\mathsf{A}^\top\mathsf{A}$ is invertible. Does your answer depend on whether $m = n$? Explain.

18. Let $\mathcal{P}_2$ be the real polynomials of degree at most two. Let

$$p_1 = 1 + t, \qquad p_2 = t + t^2, \qquad p_3 = 1 + 2t + t^2, \qquad p_4 = 1 - t + t^2.$$

    a. Choose a basis $\mathcal{B}$ for the span of $p_1, p_2, p_3, p_4$ from the given list. Describe that span and give its dimension.

    b. Find the coordinate column $[q]_{\mathcal{B}}$ for $q(t) = 2 + 3t + 4t^2$.

    c. Find every coefficient column $\boldsymbol{c} \in \mathbb{R}^4$ satisfying $c_1p_1 + c_2p_2 + c_3p_3 + c_4p_4 = q$. Are these coefficients unique? Are coordinates in the extracted basis unique? Explain.

19. For $a \in \mathbb{R}$, consider

$$\mathsf{A}_a = \begin{bmatrix} 2 & a & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

    a. Find the characteristic polynomial and the algebraic multiplicity of each eigenvalue.

    b. Find each eigenspace for every real $a$. State the geometric multiplicities and determine exactly when an eigenvector basis exists.

    c. Determine when $\mathsf{A}_a$ is invertible and find its inverse. Explain why invertibility of the matrix does not guarantee invertibility of a proposed matrix of eigenvectors.

    d. A student says, "A repeated eigenvalue prevents diagonalization." Another says, "Nonzero eigenvalues guarantee diagonalization." Use this family to assess both statements, and state the correct criterion for diagonalizability over $\mathbb{R}$.

20. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be unit vectors in a real inner product space. Their inner product is $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -1$. What is $\boldsymbol{v}$ in terms of $\boldsymbol{u}$?

21. On $\mathbb{R}^2$, consider three linear maps. The map $\boldsymbol{x} \mapsto \mathsf{R}\boldsymbol{x}$ rotates vectors counterclockwise through $90°$. The map $\boldsymbol{x} \mapsto \mathsf{H}\boldsymbol{x}$ reflects them across $y = x$. The map $\boldsymbol{x} \mapsto \mathsf{P}\boldsymbol{x}$ projects them orthogonally onto the $x$-axis.

    a. Find the matrices $\mathsf{R}, \mathsf{H}, \mathsf{P}$ in the standard basis of $\mathbb{R}^2$.

    b. Find $\mathsf{RH}$ and $\mathsf{HR}$, identify each geometric action, and give a vector demonstrating that these compositions are different.

    c. For $\mathsf{T} = \mathsf{RH}$, find $\mathsf{T}^{-1}$ and $\det(\mathsf{T})$. Explain their geometric meaning.

    d. Compute $\mathsf{C} = \mathsf{RPH}$. Describe its image and null space, and explain precisely why the original vector cannot always be recovered from $\mathsf{C}\boldsymbol{x}$.

22. Let

$$\mathsf{A} = \begin{bmatrix} 3 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}, \qquad \mathsf{U}_r = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix}, \qquad \mathsf{V}_r = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

    a. Verify that $\mathsf{U}_r$ and $\mathsf{V}_r$ have orthonormal columns. Find the diagonal matrix $\mathsf{\Sigma}_r$, with positive diagonal entries in decreasing order, for which $\mathsf{A} = \mathsf{U}_r \mathsf{\Sigma}_r \mathsf{V}_r^\top$. Describe the image of the unit circle in $\mathbb{R}^2$.

    b. Give a basis and dimension for the column space $\mathrm{col}(\mathsf{A})$, row space $\mathrm{row}(\mathsf{A})$, null space $\mathrm{null}(\mathsf{A})$, and left null space $\mathrm{null}(\mathsf{A}^\top)$. State the ambient space of each.

    c. Find a best rank-one approximation $\mathsf{A}_1$. Compute both $\|\mathsf{A} - \mathsf{A}_1\|_2$ and $\|\mathsf{A} - \mathsf{A}_1\|_F$, where the norms are the maximum unit-vector stretch and the square root of the sum of squared entries, respectively.

    d. Find the 2-norm condition number of $\mathsf{A}$ and that of $\mathsf{A}^\top\mathsf{A}$. Explain why the 2-norm condition number is defined for the rectangular matrix $\mathsf{A}$ even though an eigenvalue diagonalization of $\mathsf{A}$ is not.

23. Let $\mathsf{A}$ and $\mathsf{B}$ be invertible matrices of the same size. Express the inverse $(\mathsf{AB})^{-1}$ of the product in terms of $\mathsf{A}^{-1}$ and $\mathsf{B}^{-1}$.

24. Fit an affine function $\widehat{y}(t) = x_1 + x_2 t$ to the three observations

$$(t, y) = (0, 1),\ (1, 2),\ (3, 2)$$

by minimizing the sum of squared vertical response errors.

a. Write the design matrix $\mathsf{A}$ and observed response vector $\boldsymbol{b}$ for $\mathsf{A}\boldsymbol{x} \approx \boldsymbol{b}$, where $\boldsymbol{x} = (x_1, x_2)^\top$. Is an exact fit possible? Justify your answer.

b. Find the coefficients that minimize the stated error. Determine whether they are unique, and justify your conclusion.

c. Find the fitted response vector $\widehat{\boldsymbol{b}}$, the residual vector $\boldsymbol{r} = \boldsymbol{b} - \widehat{\boldsymbol{b}}$, and the minimum sum of squared errors. Verify that $\boldsymbol{r}$ lies in the null space of $\mathsf{A}^\top$, and explain why the entries of $\boldsymbol{r}$ sum to zero.

d. If all three responses increase by 2, find the new fitted function and residual. What distance does the least-squares projection minimize, and how does it differ from the perpendicular distances from the plotted points to the line?

25. Let

$$\mathsf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

a. Find the monic characteristic polynomial and a basis for each eigenspace. Find the trace and determinant, and explain their relationships to the eigenvalues.

b. Form an eigenvector matrix $\mathsf{V}$, ordering its columns by decreasing eigenvalue. Find $\mathsf{V}^{-1}$ and a diagonal matrix $\mathsf{D}$ such that $\mathsf{A} = \mathsf{V}\mathsf{D}\mathsf{V}^{-1}$.

c. Find the coordinates of $\boldsymbol{x}$ in your eigenvector basis. Compute $\mathsf{A}\boldsymbol{x}$.

d. Are your two eigendirections perpendicular? Can this particular matrix be diagonalized by an orthogonal change of basis? Explain how the requirements for diagonalization and orthogonal diagonalization differ.

26. To first order near $(s, t) = (2, 3)$, by what factor does $F(s, t) = (st, s)$ scale area? Does it preserve or reverse orientation there?

27. Two groups exchange population according to the supplied model

$$\boldsymbol{x}_{k+1} = \mathsf{A}\boldsymbol{x}_k, \qquad \mathsf{A} = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} 12 \\ 0 \end{bmatrix}.$$

The first and second entries of $\boldsymbol{x}_k$ give the populations of groups 1 and 2, respectively, after $k$ steps.

a. Explain why this model preserves the total population and keeps nonnegative populations nonnegative.

b. Find the eigenvalues and an eigenvector basis. Identify the direction of equilibrium and the direction whose contribution changes over time.

c. Express $\boldsymbol{x}_0$ in that basis and obtain a formula for $\boldsymbol{x}_k$ valid for every integer $k \geq 0$.

d. Find the limiting populations. Find the smallest nonnegative integer $k$ for which the second population differs from its limiting value by at most $1/4$. Does $\mathsf{A}^k$ tend to the zero matrix? Explain.

28. All spaces and scalars in this question are real. Let $\mathcal{P}_2$ denote the polynomials of degree at most two, with the usual operations.

a. Show that $W = \{p \in \mathcal{P}_2 : p(1) = 0\}$ is a subspace, and find a basis and its dimension.

b. For $F = \{p \in \mathcal{P}_2 : p(1) = 1\}$, decide whether $F$ is a subspace and show that $F = 1 + W$. Explain what stays the same when one solution of the condition is subtracted from another.

c. Decide whether $U = \{p \in \mathcal{P}_2 : p(0)p(1) = 0\}$ is a subspace. If it is not, exhibit a specific failed closure property.

d. Show that the real $2 \times 2$ matrices with trace zero form a subspace. Find a basis and its dimension, treating each whole matrix as one vector.

29. For $\mathsf{A} = \begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/4 \end{bmatrix}$, does the series of matrix powers $\sum_{k=0}^{\infty} \mathsf{A}^k$ converge? If so, find its sum.

30. Let

$$\mathsf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

a. Find the characteristic polynomial, all eigenvalues with their algebraic multiplicities, and a basis for each eigenspace.

b. Construct an orthogonal matrix $\mathsf{Q}$ and diagonal matrix $\mathsf{D}$ such that $\mathsf{A} = \mathsf{Q}\mathsf{D}\mathsf{Q}^{\top}$. Explain why any two eigenvectors belonging to distinct eigenvalues must be perpendicular here.

c. Find the coordinates of $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in your ordered orthonormal eigenbasis. Compute $\mathsf{A}\boldsymbol{x}$.

d. A real symmetric matrix is positive definite when $\boldsymbol{z}^{\top}\mathsf{A}\boldsymbol{z} > 0$ for every nonzero real $\boldsymbol{z}$. Determine whether $\mathsf{A}$ is positive definite. Find the smallest and largest possible values of $\|\mathsf{A}\boldsymbol{z}\|$ when $\|\boldsymbol{z}\| = 1$, where the norm is Euclidean, and describe directions attaining them.

31. Let $\mathcal{P}_1$ be the real polynomials of degree at most one. Use the bases $\mathcal{B} = \{1 + t, 1 - t\}$ and $\mathcal{C} = \{1, t\}$ in their listed orders. Define $T : \mathcal{P}_1 \to \mathcal{P}_1$ by $T(p) = (1 + t)p'$.

a. Find the transition matrices $\mathsf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ and $\mathsf{P}_{\mathcal{B} \leftarrow \mathcal{C}}$, explaining what their columns mean. Find both coordinate columns of $p(t) = 3 - t$.

b. Find the matrix of $T$ in basis $\mathcal{C}$, $[T]_{\mathcal{C} \leftarrow \mathcal{C}}$, and its matrix in basis $\mathcal{B}$, $[T]_{\mathcal{B} \leftarrow \mathcal{B}}$.

c. For $p(t) = 3 - t$, find the coordinate column $[T(p)]_{\mathcal{B}}$, then reconstruct the polynomial $T(p)$ and check it directly from the definition.

d. Find a basis for the kernel and for the image of $T$. Is $T$ invertible? Relate your answer to the two matrices representing $T$.

32. A zero-mean Gaussian vector has covariance matrix $\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find its precision matrix $\mathsf{Q}$ and evaluate the quadratic form $\boldsymbol{r}^{\top}\mathsf{Q}\boldsymbol{r}$ for $\boldsymbol{r} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

33. On the real polynomial space use

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let $\mathcal{P}_2$ be the real polynomials of degree at most two and let $f(t) = t^3 + 2t$.

a. Construct an ordered orthonormal basis $\{q_1, q_2, q_3\}$ of $\mathcal{P}_2$, where $q_j$ has degree $j-1$ and a positive leading coefficient. Show your work.

b. Find the affine polynomial $p(t) = a + bt$ minimizing $\int_{-1}^{1}(f(t) - p(t))^2\,dt$.

c. Verify directly that $f - p$ is orthogonal to both 1 and $t$, and compute the minimum squared error $\|f - p\|^2$ and the norm $\|f - p\|$.

d. Would allowing a quadratic polynomial give a smaller error for this particular $f$? Justify your answer and explain why the best approximation is unique.

34. Let

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

a. Find a compact singular value decomposition (SVD) of $\mathsf{A}$. Compute its Moore–Penrose pseudoinverse, denoted $\mathsf{A}^{\dagger}$.

b. Find the pseudoinverse coefficient vector $\mathsf{A}^{\dagger}\boldsymbol{b}$, the full set of least-squares coefficient vectors, the fitted vector $\widehat{\boldsymbol{b}}$, and the residual $\boldsymbol{r} = \boldsymbol{b} - \widehat{\boldsymbol{b}}$. Which of these quantities are unique?

c. Prove that the pseudoinverse coefficient vector has the smallest Euclidean norm among all the least-squares coefficient vectors.

d. Let $\boldsymbol{u}_1$, $\boldsymbol{v}_1$, and $\sigma_1$ be the left singular vector, right singular vector, and positive singular value from part a. Set

$$\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \mathsf{A}_{\varepsilon} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\top} + \varepsilon \boldsymbol{u}_2 \boldsymbol{v}_2^{\top}, \quad 0 < \varepsilon < 5.$$

If the response changes from $\boldsymbol{b}$ to $\boldsymbol{b} + \delta\boldsymbol{u}_2$, find the changes in the least-squares coefficients and fitted vector for $\mathsf{A}_{\varepsilon}$. Explain the distinction between a zero singular value and a small positive singular value.

35. Let $\mathsf{B} = \mathsf{S}^{-1}\mathsf{A}\mathsf{S}$ with $\mathsf{S}$ invertible. The eigenspace of $\mathsf{A}$ for eigenvalue 3 has dimension two. What is the dimension of the eigenspace of $\mathsf{B}$ for eigenvalue 3?

36. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}.$$

a. Give bases for the column space $\mathrm{col}(\mathsf{A})$, the row space $\mathrm{row}(\mathsf{A})$, and the null space $\mathrm{null}(\mathsf{A})$.

b. Find a basis for the left null space $\mathrm{null}(\mathsf{A}^\top)$. State the ambient space and dimension of each of the four spaces. Explain how the four dimensions are related.

c. Give a necessary and sufficient condition on $\boldsymbol{b} \in \mathbb{R}^3$ for $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ to be consistent. Prove that your condition is both necessary and sufficient.

d. Describe every solution when $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Is this solution set a subspace? State its dimension as an affine set.